Solutions 1

Exercise 2.7

$$\frac{\hat{\theta}-\theta}{\sqrt{\mathrm{var}(\hat{\theta})}} \sim \mathcal{N}(0,1) \qquad \frac{\check{\theta}-\theta}{\sqrt{\mathrm{var}(\check{\theta})}} \sim \mathcal{N}(0,1)$$

Denote cumulative distribution function $Q(x)$ as

$$Q(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

Then

$$\Pr\{|\hat{\theta}-\theta| > \epsilon\} = \Pr\{\frac{|\hat{\theta}-\theta|}{\sqrt{\mathrm{var}(\hat{\theta})}} > \frac{\epsilon}{\sqrt{\mathrm{var}(\hat{\theta})}}\} = 2Q(\frac{-\epsilon}{\sqrt{\mathrm{var}(\hat{\theta})}})$$

Since $\mathrm{var}(\hat{\theta}) < \mathrm{var}(\check{\theta})$, we can get $Q(\frac{-\epsilon}{\sqrt{\mathrm{var}(\hat{\theta})}}) < Q(\frac{-\epsilon}{\sqrt{\mathrm{var}(\check{\theta})}})$. Thus,

$$\Pr\{|\hat{\theta}-\theta| > \epsilon\} < \Pr\{|\check{\theta}-\theta| > \epsilon\}$$

Exercise 2.11

We need $\hat{\theta} = g(x[0])$ to be unbiased, i.e. $\mathbb{E}[\hat{\theta}] = \theta$. Thus,

$$\int_0^{\frac{1}{\theta}} g(x[0])f(x[0])dx[0] = \int_0^{\frac{1}{\theta}} g(x[0])\theta dx[0] = \theta$$

We can obtain

$$\int_0^{\frac{1}{\theta}} g(u)du = 1$$

Assume a function g can be found, then for any $\theta_2 < \theta_1$, we have

$$\int_0^{\frac{1}{\theta_1}} g(u)du = 1 \qquad \int_0^{\frac{1}{\theta_2}} g(u)du = 1$$

This means $\int_{\frac{1}{\theta_1}}^{\frac{1}{\theta_2}} g(u)du = 0$, for any $\theta_2 < \theta_1$. Obviously, $g(u)$ should be 0 for any $u > 0$. Hence, it doesn't exit.

Exercise 3.6

$$x[0] \sim \mathcal{N}(\theta,1) \qquad x[1] \sim \begin{cases} \mathcal{N}(\theta,1), & \theta \geq 0 \\ \mathcal{N}(\theta,2), & \theta < 0 \end{cases}$$

Then, we have

$$p(\mathbf{x};\theta) = \begin{cases} \frac{1}{2\pi} e^{-\frac{1}{2}[(x[0]-\theta)^2 + (x[1]-\theta)^2]}, & \theta \geq 0 \\ \frac{1}{2\pi\sqrt{2}} e^{-\frac{1}{2}[(x[0]-\theta)^2 + \frac{1}{2}(x[1]-\theta)^2]}, & \theta < 0 \end{cases}$$

For $\theta \geq 0$, we obtain

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = (x[0]-\theta) + (x[1]-\theta) \qquad \frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2} = -2$$

Computing CRLB, we have

$$\text{var}(\hat{\theta}) \geq \frac{1}{-\mathbb{E}[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}]} = \frac{1}{2}$$

For $\theta < 0$, we obtain

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = (x[0]-\theta) + \frac{1}{2}(x[1]-\theta) \qquad \frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2} = -\frac{3}{2}$$

Computing CRLB, we have

$$\text{var}(\hat{\theta}) \geq \frac{1}{-\mathbb{E}[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}]} = \frac{2}{3}$$

It agrees with the results given.

Exercise 3.11

Since $\mathbf{I}(\boldsymbol{\theta})$ is positive definite, we can get $a > 0$, $c > 0$ and $ac - b^2 > 0$ by $\det(\mathbf{I}(\boldsymbol{\theta})) > 0$. Besides, we can obtain

$$[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{11} = \frac{c}{ac - b^2} = \frac{1}{a - b^2/c} \geq \frac{1}{a} = \frac{1}{[\mathbf{I}(\boldsymbol{\theta})]_{11}}$$

It means the CRLB will increase when we estimate additional parameters. Equality holds when $b = 0$, because in this case the Fisher information matrix is decoupled, i.e. the additional parameter won't affect our desired parameter.

Exercise 3.12

Note that $\sqrt{\mathbf{I}^{-1}(\boldsymbol{\theta})} = (\sqrt{\mathbf{I}(\boldsymbol{\theta})})^{-1}$, we have

$$\left(\mathbf{e}_i^T \mathbf{I}(\boldsymbol{\theta}) \mathbf{e}_i\right) \cdot \left(\mathbf{e}_i^T \mathbf{I}^{-1}(\boldsymbol{\theta}) \mathbf{e}_i\right) = [\mathbf{I}(\boldsymbol{\theta})]_{ii} \cdot [\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ii} \geq \left(\mathbf{e}_i^T \sqrt{\mathbf{I}(\boldsymbol{\theta})} \sqrt{\mathbf{I}^{-1}(\boldsymbol{\theta})} \mathbf{e}_i\right)^2 = 1$$

Thus,

$$[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ii} \geq \frac{1}{[\mathbf{I}(\boldsymbol{\theta})]_{ii}}$$

The new bound will be achieved when an efficient estimation exits.

Exercise 3.13

Using the result in (3.33) of Page 49 and letting $s[n;\mathbf{A}] = \sum_{k=0}^{p-1} A_k n^k$, we can get

$$\begin{aligned}
\mathbf{I}(\mathbf{A})_{ij} &= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(\frac{\partial}{\partial A_{i-1}} \sum_{k=0}^{p-1} A_k n^k \right) \cdot \left(\frac{\partial}{\partial A_{j-1}} \sum_{k=0}^{p-1} A_k n^k \right) \\
&= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} n^{i-1} \cdot n^{j-1} \\
&= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} n^{i+j-2}
\end{aligned}$$